The M5-brane elliptic genus: modularity and BPS states

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ABSTRACT: The modified elliptic genus for an M5-brane wrapped on a four-cycle of a Calabi-Yau threefold encodes the degeneracies of an infinite set of BPS states in four dimensions. By holomorphy and modular invariance, it can be determined completely from the knowledge of a finite set of such BPS states. We show the feasibility of such a computation and determine the exact modified elliptic genus for an M5-brane wrapping a hyperplane section of the quintic threefold.

Contents

1. Introduction

An exact counting of BPS states in $\mathcal{N} = 2, d = 4$ compactifications of (say type IIA) string theory is a difficult problem. For large charges these objects have dual descriptions as supersymmetric black holes. The leading and subleading order entropies of such black holes in M-theory has been explained using the $(0,4)$ CFT on the effective string coming from a wrapped M5-brane [1]. Herein we refine the analysis of [1] and consider the modified elliptic genus (to be defined later) which counts BPS states of D4-D2-D0 system on a generic Calabi-Yau threefold. As emphasized in [2 – 5], modular invariance imposes strong constraints on the elliptic genus, and determines it completely in terms of a finite number of coefficients in its q-expansion. In this paper we will use this fact to derive the exact partition function of a basic class of BPS states in type IIA string compactified on the quintic threefold. Our results shed light on the conjectured relation [6 – 16]–[17] between the exact partition function of the black hole and the topological string amplitude.

An alternative approach - related by M/IIA duality - to count the degeneracy of D4-D2-D0 bound states is to quantize the classical moduli space of supersymmetric D-brane configurations. The latter involve wrapped D4-brane with fluxes bound to pointlike instantons. The possible U(1) fluxes on the D4-brane are in 1-1 correspondence with divisors

(an algebraic sum of holomorphic curves) on its world volume, and in particular involves nontrivial information of Gromov-Witten invariants. We use this method to compute the coefficients of the modified elliptic genus in the example of the quintic for small charges corresponding to the first few terms in the q-expansion. A priori, one might expect the results to overdetermine the elliptic genus: in principle, a few of these numbers are sufficient to determine the elliptic genus, and all other coefficients are predicted based on the modular property. This would give a sharp test of the direct counting of supersymmetric D4-D2-D0 bound states. In the simple example we consider, the system of minimal D4-brane on the quintic, this agreement requires highly nontrivial relations among Gromov-Witten invariants of various degrees. Miraculously, we find this relation to hold in all the coefficients we computed based on the geometry of the classical moduli space, up to a small ambiguity due to singularities in the moduli space, corresponding to colliding pointlike D0 instantons.

We propose an approach to resolve this ambiguity, based on the relevant dual M-theory $AdS_3 \times S^2 \times CY$ attractor geometry [16]. In this picture the needed coefficients of the q-expansion are supplied by the degeneracies of a few low-lying BPS states of gravitons and wrapped (anti-)M2-branes. It is not clear to us why this dual picture should be valid, since some of the charges are small in all of our our examples. Nevertheless, with this approach, we find perfect agreement of the degeneracy of BPS states with the relations expected from the modular property of the elliptic genus. (This in turns suggests that there should be some justification for our approach.)

This approach sheds light on the conjecture [6] that the Gromov-Witten invariants are captured by the black hole partition function. Here we see that the $U(1)$ fluxes on the D4-brane, which are related to curves in the CY subject to the constraint that it must lie on the world volume of the D4-brane, carry information about Gromov-Witten invariants. This extra constraint becomes unimportant for large D4-brane charges (a sufficiently high degree hypersurface can be made to pass through any given collection of curves), in which limit the (square of the) topological string partition function becomes a good approximation of the black hole partition function.

In section 2, we describe the general structure of the modified elliptic genus of the MSW $(0,4)$ CFT. We argue that the elliptic genus has simple anti-holomorphic dependence, and can be determined by a finite number of holomorphic characters that transform in a known representation of $SL(2,\mathbf{Z})$. In section 3, we study the $(0,4)$ CFT for an M5-brane wrapped on the hyperplane section in the quintic. We count the degeneracy of a number of BPS states based on the classical moduli space of D-brane configurations, as well as a hypothetical chiral ring structure motivated by a dilute gas approximation in the AdS_3 dual. These results are compared to the modular property of the elliptic genus, and surprising agreements are found. In particular, we conjecture an exact expression for the elliptic genus in this case. Section 4 studies a different example, a free $\mathbf{Z}_5$ quotient of the Fermat quintic. In this case we present the structure of the elliptic genus as determined by its modular property, although the direct counting based on the classical moduli space is more difficult than the quintic, and is left to future work. We conclude in section 5.

Results related to those of section 2 on the structure of the modified elliptic genus have been independently obtained by Denef and Moore [5], and de Boer, Cheng, Dijkgraaf,

Manschot and E. Verlinde [4] .

2. The M5-brane $(0,4)$ CFT

We will be considering an M-theory 5-brane wrapped on a 4-cycle P in Calabi-Yau space X, and extended in $\mathbf{R}^{1,4}$. The low energy excitations of the M5-brane can be described by an effective $1 + 1$ dimensional CFT [1], which has $(0,4)$ superconformal symmetry. If one further compactifies the direction in which the M5-brane extends in $\mathbf{R}^{1,4}$ on a circle, one obtains a wrapped D4-brane in type IIA string theory compactified on X. Excitations of the M5-brane induce M2-brane charges and can carry momenta. These in general correspond to D4-D2-D0 bound states in type IIA string theory. The correspondence between M5-brane states and D4-D2-D0 bound states is understood and will be used freely in this paper.

In the following, we will follow the convention of [1] and denote by $6D_{ABC}$ the intersection numbers in a basis Σ_A of $H_4(X, \mathbf{Z})$. The D4, D2, D0 charges will often be labelled p^A, q_A, q_0, respectively. $D_{AB} \equiv D_{ABC}p^C$, $D \equiv D_{ABC}p^Ap^Bp^C$, and D^{AB} is the inverse matrix of D_{AB}. The attractor Kähler class of X is proportional to $J = p^A\omega_A$ where ω_A is a basis of harmonic $(1,1)$ forms dual to Σ_A.

2.1 General structure

The M5-brane $(0,4)$ CFT [1] has central charge $c_L = 6D + c_2 \cdot P$, $c_R = 6D + \frac{1}{2}c_2 \cdot P$. There are 3 noncompact bosons X^i on the left and right, corresponding to the collective coordinates in the transverse $\mathbf{R}^3$. There are free bosons ϕ_A, coming from the world volume anti-symmetric tensor field reduced on ω_A. Namely

$$T \sim D^{AB}d\phi_A \wedge \omega_B \tag{2.1}$$

The self-duality of T implies that $\varphi = p^A\phi_A$ is purely right moving. Together with four goldstinos $\tilde{\psi}^{\pm\pm}$, corresponding to the four supersymmetries broken by the M5-brane, $(\overline{\partial}X^i, \overline{\partial}\varphi, \tilde{\psi}^{\pm\pm})$ form a right moving $\mathcal{N} = 4$ multiplet. The ϕ_A's are compactified on a lattice of signature $(h^{1,1}(X) - 1, 1)$, with the intersection pairing given by $-\frac{1}{6}D^{AB}$, where the $(h^{1,1} - 1)$ bosons orthogonal to φ are left movers.

This CFT in fact has $\mathcal{A}_{k^+,\infty}$ symmetry algebra $(k^+ = \frac{c_R}{6})$, which is a Wigner contraction of the large $\mathcal{N} = 4$ superconformal algebra. Writing $U = \overline{\partial}\varphi$, in addition to the small $\mathcal{N} = 4$ SCA relations, there are OPEs

$$\overline{G}^{\alpha a}(\overline{z})\, U(0) \sim \frac{\tilde{\psi}^{\alpha a}(0)}{\overline{z}}$$
$$\overline{G}^{\alpha a}(\overline{z})\tilde{\psi}^{\beta b}(0) \sim \frac{\epsilon^{\alpha\beta}\epsilon^{ab}\, U(0)}{\overline{z}}$$
$$J_R^i(\overline{z})\tilde{\psi}^{\alpha a}(0) \sim \frac{(\sigma^i)^{\alpha}{}_{\beta}\tilde{\psi}^{\beta a}(0)}{\overline{z}} \tag{2.2}$$

We will restrict to the case where X is a Calabi-Yau manifold of full SU(3) holonomy. Let $\Lambda = H^2(P, \mathbf{Z})$ be the cohomology lattice of P. In general P may not be a spin manifold, and Λ, which is self-dual, may not be even. In this case there is Freed-Witten anomaly [18],

which requires one to turn on a half-integral flux $\frac{1}{2}c_1(P)$ in addition to integral fluxes on the M5-brane world volume. We will write $J = p^A \omega_A$ to represent the class of P itself, or the cohomology class dual to $P \cap P$ in $H_2(P)$, which is the same as $-c_1(P)$. One can write a modular invariant theta function associated with the lattice Λ,

$$\Theta_\Lambda(\tau, \overline{\tau}, y) = \sum_{v \in \Lambda + \frac{1}{2}J} (-)^{p \cdot q(v)} e^{-\pi i \tau v_-^2 - \pi i \overline{\tau} v_+^2 + 2\pi i q(v) \cdot y} \tag{2.3}$$

Here v_+ and v_- are the self-dual and anti-self-dual projections of lattice vector v, or equivalently, projections along J and $J^\perp$. $v_\pm^2$ are defined using the intersection form on Λ. $q(v)$ is the natural projection of v from $H_2(P, \mathbf{Z})$ to $H_2(X, \mathbf{Z})$, corresponding to the M2/D2-brane charge. Note that J is an characteristic element of $H^2(P, \mathbf{Z})$, hence $v^2 + q(v) \cdot p \equiv 0$ mod 2, $(-)^{p \cdot q(v)} = 1$ when Λ is even.

$\Lambda_X = H^2(X, \mathbf{Z})$ is embedded in Λ as a sublattice. It consists of charge vectors with $q_A = 6D_{AB}k^B$, $k^A \in \mathbf{Z}$. $\Lambda_X^\perp \subset \Lambda$ consists of only left-moving lattice vectors. The theta function (2.3) can be decomposed as

$$\Theta_\Lambda(\tau, \overline{\tau}, y) = \sum_\delta \Theta_{\Lambda_X^\perp + \delta}(\tau) \Theta_{\Lambda_X + \delta}(\tau, \overline{\tau}, y) \tag{2.4}$$

where δ runs through a finite set of $\det(6D_{AB})$ shift vectors. The functions $\Theta_{\Lambda_X + \delta}$, which we will abbreviate as Θ_δ, can be written explicitly

$$\Theta_\delta(\tau, \overline{\tau}, y) = \sum_{q_A = 6D_{AB}(k^B + \frac{1}{2}p^B) + \delta_A, k^A \in \mathbf{Z}} (-)^{p^A q_A} \exp\left[\frac{2\pi i \tau}{12} \left(\frac{p^A p^B}{D} - D^{AB} \right) q_A q_B \right. \tag{2.5}$$

$$\left. - \frac{2\pi i \overline{\tau}}{12D}(p^A q_A)^2 + 2\pi i y^A q_A \right]$$

2.2 The modified elliptic genus

We are interested in computing a supersymmetric index of the $(0,4)$ CFT. The elliptic genus vanishes, due to the degeneracy of Ramond ground state generated by the zero modes of the goldstinos $\psi_0^{\pm\pm}$. As explained in the previous subsection, the CFT has symmetry algebra $\mathcal{A}_{k^+, \infty}$, which extends the small $\mathcal{N} = 4$ superconformal algebra. One can define a modified elliptic genus [19],

$$Z(\tau, \overline{\tau}, y) = \text{Tr}_R \frac{1}{2} F^2 (-)^F q^{L_0 - \frac{c_L}{24}} \overline{q}^{\overline{L}_0 - \frac{c_R}{24}} e^{2\pi i y^A Q_A} \tag{2.6}$$

where $q = e^{2\pi i \tau}$, Q_A are the charges associated with the free bosons ϕ_A, corresponding to induced M2-brane charges on the M5-brane. F is a fermion number, which can be identified with $2J_R^3 + p^A Q_A$, where J_R is the right moving R-charge, and $p^A Q_A$ is the contribution from the quantization of the self-dual 3-form field (this is an example of a general phenomenon explained in [3]).

An important point is that not only the right moving ground states contribute to Z. Let $|0\rangle$ be a ground state, then $\tilde{\psi}_0^{\pm\pm}$ acting on $|0\rangle$ generates a multiplet of 4 states, contributing 1 to $\text{Tr}\frac{F^2}{2}(-)^F$. These states are annihilated by the supercharges $\overline{G}_0^{\pm\pm}$. Now

consider a state of charge q_A under the current $j_A = d\phi_A$, say $|q\rangle = e^{iD^{AB}q_A\phi_B}|0\rangle$. $\overline{G}_0^{\pm\pm}$ acting on $p^A\phi_A$ gives rise to its fermionic partners $\tilde{\psi}_0^{\pm\pm}$, and leave the left moving fields invariant. Hence

$$(\overline{G}_0^{\pm\pm} - p^A Q_A \tilde{\psi}_0^{\pm\pm})|q\rangle = 0 \tag{2.7}$$

This is simply saying that the state $|q\rangle$ preserves supersymmetries nonlinearly, and acting with $\overline{G}_0^{\pm\pm}$ doesn't give rise to new states other than the multiplet generated by $\tilde{\psi}_0^{\pm\pm}$. This is in accord with the fact that q_A correspond to induced D2-brane charges, and D4-D2 bound states preserve different sets of supersymmetries than that of D4(-D0). Therefore this multiplet also contributes 1 to the modified elliptic genus. In general the modified elliptic genus Z receives contribution from states of charge q_A with

$$(\overline{L}_0 - \frac{c_R}{24})|\psi\rangle = \frac{(p^A Q_A)^2}{12D}|\psi\rangle \tag{2.8}$$

As in [9, 5, 4], the modified elliptic genus has the general form

$$Z(\tau, \overline{\tau}, y) = \sum_\delta Z_\delta(\tau)\Theta_\delta(\tau, \overline{\tau}, y) \tag{2.9}$$

where Θ_δ are given by (2.5). This structure can be argued using the fact that shifting B-field by an integral amount on X does not change the degeneracy of D4-D2-D0 bound states, but generates additional D2 and D0-brane charges, corresponding to a translation in Λ_X. Another way to think about (2.9) is that the theta function of the cohomology lattice vectors of $H^2(P, \mathbf{Z})$ that do not correspond to conserved charges get completed into holomorphic characters $Z_\delta(\tau)$ in the full CFT.

$\Theta_\delta(\tau, \overline{\tau}, y)$ form a modular representation in terms of weight $(\frac{1}{2}(h^{1,1}(X)-1), \frac{1}{2})$ Jacobi forms. The T transformation is represented by the matrix

$$T_{\delta\lambda}^\Theta = \delta_{\delta\lambda} \exp\left(-\frac{2\pi i}{12}D^{AB}\delta_A\delta_B\right) \tag{2.10}$$

The S transformation is represented by

$$S_{\delta\lambda}^\Theta = \frac{1}{\sqrt{6D}}\exp\left(-\frac{2\pi i}{6}D^{AB}\delta_A\lambda_B\right) \tag{2.11}$$

The modified elliptic genus of the MSW $(0,4)$ CFT is expected to be a weight $(-\frac{3}{2}, \frac{1}{2})$ Jacobi form. The left weight $-\frac{3}{2}$ comes from the three noncompact bosons, and the right weight is modified by the insertion of F^2 to $-\frac{3}{2} + 2 = \frac{1}{2}$. $Z_\delta(\tau)$ transform under SL(2,$\mathbf{Z}$) with $T = (T^\Theta)^*$, $S = (S^\Theta)^*$ up to an overall phase that is easy to determine.

Knowing the modular representation of $Z_\delta(\tau)$, one can determine all of them from the polar terms in the q-expansion of Z_δ, where $q = e^{2\pi i\tau}$, via the generalized Rademacher expansion. Equivalently, there is a basis of modular vectors transforming the same way as $Z_\delta(\tau)$ with the most singular polar term $q^{-\frac{c_L}{24}}$, whose number is the same as the number of possible polar terms of $Z_\delta(\tau)$. We will explore the constraints of modular invariance of Z on the degeneracy of BPS states in the rest of this paper.

3. BPS states on the quintic

In this section X will be the quintic 3-fold with a generic complex structure. We will study the $(0,4)$ CFT associated with an M5-brane wrapped on the hyperplane section P in X. J will refer to the hyperplane class. This CFT has $c_L = 55, c_R = 30$, and $D = \frac{5}{6}$.

3.1 A naive counting from geometry

Our strategy will be to count the D4-D2-D0 bound states of given charges $(p = 1, q_1, q_0)$ by computing the Euler character of the classical moduli space of the branes. The D4-brane is wrapped on the hyperplane section P, and it is free to move in its moduli space $\mathbf{P}^4$. In a supersymmetric configuration, the D2-branes are dissolved into fluxes on the D4 world volume. We will assume that D0-branes can be either pointlike instantons (which can form bound states among themselves) on the D4-brane world volume, or dissolve into smooth U(1) fluxes.

One may attempt to describe the moduli space of classically supersymmetric D4-D2-D0 configuration as a fibration over the D4 moduli space, the fiber being the Hilbert scheme of points on the D4-brane world volume etc. This is a useful approximation in the limit of large D0-brane charges, but is difficult to apply for small charges. The reason is that the D4-brane world volume degenerates in various loci in its moduli space. It turns out that for the cases we will be computing, it is more useful to describe the moduli space by first fixing the D0-branes, and consider the space of D4-branes that pass through these D0-branes and admit certain classes of fluxes.

Due to Freed-Witten anomaly one must turn on half integral flux, say $F = \frac{1}{2}J$, on the D4-brane. We will call this the "pure" D4-brane. There is induced D2-brane charge $\frac{5}{2}$ and D0-brane charge [20, 21]

$$-\frac{\chi(P)}{24} - \frac{1}{2}\int_P F \wedge F = -\frac{35}{12}$$

In the CFT this corresponds to a state with $L_0 = 0$ and $(\overline{L}_0 - \frac{c_R}{24}) = \frac{(p^A q_A)^2}{12D} = \frac{5}{8}$. In the following we will label states by their additional D2-brane charge Δq_1, as well as the additional D0-brane charge Δq_0.[1]

- $\Delta q_1 = 0, \Delta q_0 = 0$ $(L_0 = 0)$

  This is the "pure" D4-brane, which is free to move around its moduli space $\mathbf{P}^4$. The number of supersymmetric states is $\chi(\mathbf{P}^4) = 5$.

- $\Delta q_1 = 0, \Delta q_0 = 1$ $(L_0 = 1)$

  Next we consider the D4 bound to a single D0-brane. Requiring the D4-brane to pass through the D0, the moduli space of D4-D0 is $\mathbf{P}^3$ fibered over X. It has Euler character $\chi(\mathbf{P}^3)\chi(X) = -800$.

- $\Delta q_1 = 0, \Delta q_0 = 2$ $(L_0 = 2)$

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[1] Δq_0 is related to L_0 and $\overline{L}_0$ by $\Delta q_0 - \frac{35}{12} = (L_0 - \frac{c_L}{24}) - (\overline{L}_0 - \frac{c_R}{24}) = (L_0 - \frac{55}{24}) - \frac{1}{10}(\Delta q_1 + \frac{5}{2})^2$.

Let us consider the D4 bound to 2 D0's. The two D0-branes can either both be free to move on the D4-brane world volume, or bind together as a single pointlike object. The latter is counted the same way as the $\Delta q_0 = 1$ case, whereas the former is described by the moduli space as $\mathbf{P}^2$ fibered over $\mathrm{Sym}^2(X)$. We shall ignore the subtle contribution from the locus in the moduli space where the two D0's coincide. The number of states, counted with sign, is then $\chi(\mathbf{P}^2)\chi(\mathrm{Sym}^2(X)) + \chi(\mathbf{P}^3)\chi(X) = 58900$.

- $\Delta q_1 = 0, \Delta q_0 = 3$ $(L_0 = 3)$

  The states with $\Delta q_0 = 3$ involves the D4 bound to 3 pointlike instantons, as well as a D4-brane with flux $F = C_1 - C_1'$, where C_1 and C_1' are two different degree 1 rational curves in X that lie in P, we write them for their dual harmonic forms on P. The latter can happen only when P passes through both C_1 and C_1'. C_1 and C_1' generically do not touch, and each have self-intersection number $C_1 \cdot C_1 = -3$.[2] The flux F gives rise to induced D0-brane charge $-\frac{1}{2}F^2 = 3$. Generically this condition fixes a unique hyperplane, and we have 2875×2874 choices of $C_1 - C_1'$. These give rise to (again, ignoring the subtlety where the D0's coincide in the moduli space)

  $$\chi(\mathbf{P}^1)\chi(\mathrm{Sym}^3(X)) + \chi(\mathbf{P}^2)\chi(X)^2 + \chi(\mathbf{P}^3)\chi(X) + 2875 \cdot 2874 = 5755150 \quad (3.1)$$

  states.

- $\Delta q_1 = 1, \Delta q_0 = 1$ $(L_0 = \frac{8}{5})$

  Next we consider the D4 bound to a single D2-brane. This can be realized by turning on a flux $F = C_1$ on the D4-brane world volume (in addition to the original $\frac{J}{2}$), where C_1 is (dual to) a degree 1 rational curve. It gives rise to additional induced D0-brane charge $-\frac{1}{2}(F + \frac{J}{2})^2 + \frac{5}{8} = 1$. Requiring the D4-brane to pass through the curve C_1 reduces its moduli space from $\mathbf{P}^4$ to $\mathbf{P}^2$. There are 2875 degree 1 rational curves C_1. Our naive quantization of the classical moduli space doesn not determine the overall fermion number of these states. However, we can fix the fermion number by comparison with the holographic dual, and these states turn out to have an odd fermion number.[3] This description will be explored in the next section. In the end, we get the counting $\chi(\mathbf{P}^2) \cdot (-2875) = -8625$.

  Similarly, the same degeneracy applies to the states with $\Delta q_1 = -1, \Delta q_0 = 2$ $(L_0 = \frac{8}{5})$.

- $\Delta q_1 = 1, \Delta q_0 = 2$ $(L_0 = \frac{13}{5})$

  Such states involve a D4 with flux $F = C_1$ where C_1 is a degree 1 rational curve, as well as a pointlike D0 instanton. We don't understand precisely what happens when the D0-brane coincides with C_1, and will again ignore this subtlety for now. When

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[2]This follows from the adjunction formula, for a curve C in P, $C \cdot C + C \cdot J = 2g - 2$.

[3]A D4-brane with flux $F = C_1$ corresponds to an M2-brane wrapped on $C_1 \subset X$ in the dual $AdS_3 \times S^2 \times X$ geometry. The chiral primary state associated with this wrapped M2-brane is a fermion.

the D0-brane is away from C_1, the D4-brane that passes through both the D0-brane and the curve C_1 has moduli space $\mathbf{P}^1$. The number of states counted with sign is

$$(-2875) \cdot \chi(X)\chi(\mathbf{P}^1) = 1150000.$$

- $\Delta q_1 = 2, \Delta q_0 = 1$ ($L_0 = \frac{12}{5}$)

  The states with D2-brane charge $\Delta q_1 = 2$ has a minimal D0-brane charge 1, coming from the D4-brane with flux $F = C_2$, where C_2 is a degree 2 rational curve. There are 609250 such curves. Hyperplanes that pass though a degree 2 curve have moduli space $\mathbf{P}^1$. We have degeneracy[4] $(-609250) \cdot \chi(\mathbf{P}^1) = -1218500$.

- $\Delta q_1 = 2, \Delta q_0 = 2$ ($L_0 = \frac{17}{5}$)

  The counting of such states receives three kinds of contributions: a D4 with flux $F = C_1 + C_1'$ where C_1, C_1' are degree 1 rational curves; D4 with flux $F = C_2$ where C_2 is a degree 2 rational curve, bound to a D0 pointlike instanton; and D4 with flux $F = J - C_3$ where C_3 is a degree 3 rational curve. The counting is (again, ignoring the subtle case when the D0 coincides with C_2)

  $$\frac{1}{2} 2875 \cdot 2874 + 2875 \cdot \chi(\mathbf{P}^2) + (-609250) \cdot \chi(X) + 317206375 = 443196375.$$

  One might think that there is another contribution, coming from D4 with flux $F = J - C_3'$ bound to a single D0, where C_3' is a degree 3 *genus 1* curve. These however give rise to the same set of fluxes as $F = C_2$.[5]

3.2 Connection to topological strings

The M5-brane $(0, 4)$ CFT is dual to M-theory on $AdS_3 \times S^2 \times X$ attractor geometry [22, 16]. It was shown in [16] that the elliptic genus that counts the chiral primaries coming from supergravity modes as well as CY-wrapped M2 and anti-M2 branes in the dilute gas approximation reproduces the square of the topological string partition function on X. The supergravity picture is not expected to be generally valid for small M5-brane charges/fluxes, although some quantities may be BPS protected. However there appears to be a rough correspondence between supersymmetric ground states of the D4-brane bound to D2, D0-branes and multi-particle chiral primaries in AdS_3:

$$\text{pointlike D0 instantons} \longleftrightarrow \text{massless supergravity modes} \tag{3.2}$$

$$\text{flux } F = \sum_i C_i - \sum_j C_j' \longleftrightarrow \text{M2 wrapped on } C_i, \quad \overline{\text{M2}} \text{ wrapped on } C_j' \tag{3.3}$$

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[4]The sign, once again, is determined by comparison to the AdS_3 dual.

[5]To see whether $F_1 = C_2$ and $F_2 = J - C_3'$ are the same flux on P, one simply needs to check whether $(F_1^- - F_2^-)^2 = 0$, where F^- is the anti-self-dual projection of F, $F^- = F - \frac{1}{5}(F \cdot J)J$. This is the case if and only if $C_2 \cdot C_3' = 6$. In fact, any C_3' can be defined by the equations of the form $P_3(x^i) = 0, H_1(x^i) = G_1(x^i) = 0$, where P_3 is a cubic polynomial in homogeneous coordinates x^i on $\mathbf{P}^4$, and H_1, G_1 are linear polynomials. One can take H_1 to be the hyperplane section P. The quintic equation must be of the form $P_3(x^i)P_2(x^i) + H_1(x^i)Q_4(x^i) + G_1(x^i)R_4(x^i) = 0$. Now $P_2(x^i)$ together with H_1, G_1 define a degree 2 rational curve C_2, which touches C_3' at 6 points. Indeed, there are 609250 degree 3 genus 1 curves in the quintic, the same as the number of degree 2 genus 0 curves.

where C_i, C'_j are holomorphic curves in X.

Under the spectral flow of $\mathcal{A}_{k^+,\infty}$ algebra from NS to Ramond sector, which includes $L_0 \rightarrow L_0 - J_R^3 + \frac{c_R}{24}$ and a shift of the membrane charges $Q_A \rightarrow Q_A + 3D_{AB}p^B$ (see also [4]), the chiral primaries flow to Ramond sector states of the form (2.7), (2.8). The unitarity bound on the chiral primaries takes the form $\overline{L}_0 = J_R^3 + \frac{(p^A Q_A)^2}{12D}$. In particular, since $(J_R)_1^-$ flows to $(J_R)_0^-$, the chiral primaries are annihilated by $(J_R)_1^-$, and flow to lowest $SU(2)_R$ weight states in the Ramond sector. For example, the AdS_3 vacuum flows to a lowest weight state of spin $J_R^3 = -\frac{c_R}{12}$. Together with the states obtained by acting with $\tilde{\psi}^{+\pm}$, this $SU(2)_R$ multiplet contributes to the modified elliptic genus with degeneracy $2(j_R - \frac{1}{2}) + 1 = \frac{1}{6}c_R = D + \frac{1}{12}c_2 \cdot P$ (with the insertion of $\frac{1}{2}F^2$ in (2.6) absorbed). This is precisely the Euler character of the moduli space of a "pure" D4-brane of charge p^A, namely $\mathbf{P}^{D+\frac{1}{12}c_2 \cdot P - 1}$.

The prescription to compute the elliptic genus is

$$Z(\tau, \overline{\tau}, y) = \text{Tr}_{ch.pr.}(-)^F \left(\frac{c_R}{6} - 2J_R^3 \right) q^{L_0 - \frac{c_L}{24}} \overline{q}^{\frac{(p \cdot Q')^2}{12D}} e^{2\pi i y^A Q'_A} \tag{3.4}$$

where $Q'_A \equiv Q_A + 3D_{AB}p^B$, and $(\frac{c_R}{6} - 2J_R^3)$ is the contribution due to the $SU(2)_R$ multiplets described above.

We propose to compute the first few terms in the elliptic genus using the dilute gas approximation in the AdS_3. This involves a free gas of massless supergravity modes and wrapped M2 and anti-M2-branes. They can carry angular momenta on the S^2 as well as in the AdS_3. In this approach, one can determine the fermion number of the chiral primaries corresponding to the wrapped M2-branes, as in [16], which is hard to determine by directly quantizing the classical moduli space of D4-branes.

Let us denote by $\mathcal{O}_{n,j}^a$ the chiral primary operator dual to a graviton/hyper/vector multiplet of spin j on the S^2, and $L_0 - \overline{L}_0 = n$; and $\mathcal{O}_{n,j}^C$ ($\mathcal{O}_{n,j}^{-C}$) the chiral primaries dual to M2-brane (anti-M2-brane) carrying spin j and $L_0 - \overline{L}_0 = n$.[6] The L_{-1} descendants of the chiral primaries are dual to holomorphic derivatives of the corresponding operators. We have $n = \frac{1}{2}$ for the massless hypermultiplets, $n = -1, 0, 1, 2$ for the graviton multiplet, and $n = 0, 1$ for the vector multiplets. And $n = \frac{1}{2}$ for an (anti-)M2-brane wrapped on a rational curve [16]. In the case of the quintic, the contribution to the elliptic genus from 204 hypermultiplets, 1 vector multiplet and 1 graviton multiplet is equivalent to 200 hypermultiplets.

In the following we redo the counting in the previous subsection using this dilute gas approximation (in the chiral ring language), which does not have the ambiguity with singularities of the classical moduli space of the D4-brane.

- $q_1 = 0, L_0 = 1$

  There are 200 $\mathcal{O}_{\frac{1}{2},\frac{1}{2}}^a$'s (counted with sign) that contribute, each giving rise to a fermionic Ramond state of spin $j_R = 2 - \frac{1}{2} = \frac{3}{2}$. Hence the contribution to the

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[6]It would be useful to justify the chiral ring generators and (lack of) relations directly from the sigma model description of the $(0,4)$ CFT [23].

(modified) elliptic genus is $-4 \times 200 = -800$. This is the same answer as the one we obtained from the "naive" counting.

- $q_1 = 0, L_0 = 2$

  The states that contribute are given by operators of the form $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^b_{\frac{1}{2},\frac{1}{2}}$, $\partial\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}$, $\mathcal{O}^a_{\frac{1}{2},\frac{3}{2}}$, of spin $1, \frac{1}{2}$ and $\frac{3}{2}$ respectively. The counting is then

  $$3 \cdot \left(\frac{1}{2}200 \cdot 199\right) - 4 \cdot 200 - 2 \cdot 200 = 58500$$

  Note that this differs slightly from the answer 58900 we obtained from the "naive" counting.

- $q_1 = 0, L_0 = 3$

  The states that contribute are given by operators of the form $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^b_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^c_{\frac{1}{2},\frac{1}{2}}$, $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^b_{\frac{1}{2},\frac{3}{2}}$, $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\partial\mathcal{O}^b_{\frac{1}{2},\frac{1}{2}}$, $\mathcal{O}^a_{\frac{1}{2},\frac{5}{2}}$, $\partial\mathcal{O}^a_{\frac{1}{2},\frac{3}{2}}$, $\partial^2\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}$, $\mathcal{O}^{C_1}_{\frac{1}{2},1}\mathcal{O}^{-C_1'}_{\frac{1}{2},1}$. The counting is

  $$-2 \cdot (200 \cdot 199 \cdot 198/6) + (3+1) \cdot 200 \cdot 200 - (4+2+0) \cdot 200 + 2875^2 = 5797625$$

- $q_1 = 1, L_0 = \frac{3}{2} + \frac{q^2}{10} = \frac{8}{5}$

  The operators that contribute are $\mathcal{O}^{C_1}_{\frac{1}{2},1}$, which gives rise to $-3 \times 2875 = -8625$ states, the same as the naive counting in the previous section.

- $q_1 = 1, L_0 = \frac{13}{5}$

  The operators that contribute are $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^{C_1}_{\frac{1}{2},1}$, $\mathcal{O}^{C_1}_{\frac{1}{2},2}$, $\partial\mathcal{O}^{C_1}_{\frac{1}{2},1}$. The counting is

  $$2 \cdot (-200) \cdot (-2875) + (3+1) \cdot (-2875) = 1138500$$

- $q_1 = 2, L_0 = 2 + \frac{q^2}{10} = \frac{12}{5}$

  The operators that contribute are $\mathcal{O}^{C_2}_{\frac{1}{2},\frac{3}{2}}$. The counting is the same as before, giving rise to -1218500 states.

- $q_1 = 2, L_0 = \frac{17}{5}$

  The operators that contribute are $\mathcal{O}^a_{\frac{1}{2},\frac{1}{2}}\mathcal{O}^{C_2}_{\frac{1}{2},\frac{3}{2}}$, $\mathcal{O}^{C_2}_{\frac{1}{2},\frac{5}{2}}$, $\partial\mathcal{O}^{C_2}_{\frac{1}{2},\frac{3}{2}}$, $\mathcal{O}^{C_1}_{\frac{1}{2},1}\mathcal{O}^{C_1'}_{\frac{1}{2},1}$, $(\mathcal{O}^J\mathcal{O}^{-C_3})_{1,2}$.[7] The counting is then

  $$(-200) \cdot (-609250) + (2+0) \cdot (-609250) + 2875 \cdot 2874/2 + 317206375 = 441969250.$$

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[7]Here $\mathcal{O}^J$ is a kind of spectral flow operator that shifts $Q_A \to Q_A + 6D_{AB}p^B$ (see also [4]). In particular, it shifts the $\overline{L}_0$ value of $\mathcal{O}^{-C_3}_{\frac{1}{2},2}$ by $\frac{(-3+5)^2}{10} - \frac{(-3)^2}{10} = -\frac{1}{2}$, and hence the resulting operator $\mathcal{O}^J\mathcal{O}^{-C_3}$ has $L_0 - \overline{L}_0 = 1$. This operator is reminiscent of the state coming from the D4-brane with flux $F = J - C_3$ in the previous subsection.

3.3 Constraints from modularity

Based on the general structure of the modified elliptic genus (2.9), we can write for M5-brane with charge $p = 1$ on the quintic X,

$$Z(\tau, \overline{\tau}, y) = \sum_{k=0}^{4} Z_k(q) \Theta_k(\overline{q}, z) \tag{3.5}$$

where

$$\Theta_k(\overline{q}, z) = \sum_n (-)^{n+k} \overline{q}^{\frac{1}{2}5(n+\frac{k}{5}+\frac{1}{2})^2} z^{5n+k+\frac{5}{2}}, \quad z = e^{2\pi i y}. \tag{3.6}$$

Using the results of our naive counting of BPS states based on the classical geometry of D4 with fluxes, we can write the first few terms of the q-expansion of the functions (Z_0, Z_1, Z_2),

$$Z_0^{cl}(q) = q^{-\frac{55}{24}}(5 - 800q + 58900q^2 + 5755150q^3 + \cdots)$$
$$Z_1^{cl}(q) = Z_4^{cl}(q) = q^{-\frac{55}{24}+\frac{3}{5}}(8625q - 1150000q^2 + \cdots)$$
$$Z_2^{cl}(q) = Z_3^{cl}(q) = q^{-\frac{55}{24}+\frac{2}{5}}(-1218500q^2 + 443196375q^3 + \cdots) \tag{3.7}$$

Alternatively, counting gravitons and wrapped (anti-)M2-branes in AdS_3 in the dilute gas approximation gives

$$Z_0^{c.r.}(q) = q^{-\frac{55}{24}}(5 - 800q + 58500q^2 + 5797625q^3 + \cdots)$$
$$Z_1^{c.r.}(q) = Z_4^{c.r.}(q) = q^{-\frac{55}{24}+\frac{3}{5}}(8625q - 1138500q^2 + \cdots)$$
$$Z_2^{c.r.}(q) = Z_3^{c.r.}(q) = q^{-\frac{55}{24}+\frac{2}{5}}(-1218500q^2 + 441969250q^3 + \cdots) \tag{3.8}$$

Under $SL(2, \mathbf{Z})$, the Θ_k's transform as

$$(T\Theta_k)(\overline{q}, z) = e^{-\pi i (\frac{k}{5}+\frac{1}{2})^2} \Theta_k(\overline{q}, z),$$
$$(S\Theta_k)(\overline{q}, z) = \sum_{l=0}^{4} e^{-\frac{2\pi i}{5}kl} \Theta_l(\overline{q}, z) \tag{3.9}$$

It follows that (Z_0, Z_1, Z_2) form a modular representation, with T-transformation

$$T = e^{-2\pi i \frac{55}{24}} \begin{pmatrix} 1 & & \\ & \omega^3 & \\ & & \omega^2 \end{pmatrix} \tag{3.10}$$

and S-transformation

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \omega + \omega^4 & \omega^2 + \omega^3 \\ 1 & \omega^2 + \omega^3 & \omega + \omega^4 \end{pmatrix} \tag{3.11}$$

Knowing the T and S matrix, one can determine (Z_0, Z_1, Z_2) solely from their polar terms by the generalized Rademacher expansion. In our case, however, it is possible to identify a basis of the exact modular forms, and hence determining (Z_0, Z_1, Z_2) using any four of the coefficients in their q-expansions to fix the three functions completely. This would

provide a highly nontrivial check on our "naive" counting of BPS states (3.7), and the improved (3.8).

As explained in the appendix, it is possible to construct three sets of modular vectors $(A_k(\tau)), (B_k(\tau)), (C_k(\tau))$, $k = 0, \ldots, 4$, of weight $2, 4, 6$ respectively, that transform under $SL(2, \mathbf{Z})$ with the same T and S matrix as those of $(Z_k(\tau))$ up to an overall phase, and have no polar terms. A basis for weight $-\frac{3}{2}$ modular vectors in the same representation as Z_k and with the appropriate polar terms is given by

$$\eta^{-55} A_k E_4^6, \eta^{-55} A_k E_4^3 E_6^2, \eta^{-55} A_k E_6^4, \eta^{-55} B_k E_4^4 E_6, \eta^{-55} B_k E_4 E_6^3, \eta^{-55} C_k E_4^5, \eta^{-55} C_k E_4^2 E_6^2.$$
(3.12)

where E_4 and E_6 are Eisenstein series. This basis consists of modular forms that involve $q^{-\frac{55}{24} + \frac{1}{10} + \frac{1}{2}}$ term in Z_1 and $q^{-\frac{55}{24} + \frac{2}{5}}$, $q^{-\frac{55}{24} + \frac{2}{5} + 1}$ terms in Z_2, which are clearly absent by examining the allowed D-brane charges in the supersymmetric bound states. Taking these terms into account, we have 7 possible polar terms in (Z_0, Z_1, Z_2), which exactly match the basis of 7 modular vectors (3.12).

It turns out that (3.7) almost fits in the q-expansion of the exact modular forms, but has about 1% error in some of the coefficients. It seems clear that the numbers $5, -800, 8625 = 3 \times 2875$ and $-1218500 = -2 \times 609250$ are obtained in unambiguous ways, whereas in counting the other coefficients in (3.7) we ignored the subtlety when the D0-branes coincide etc. By fitting the former four numbers with the basis (3.12), we find the exact modular vectors

$$Z_0(q) = q^{-\frac{55}{24}}(5 - 800q + 58500q^2 + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 \cdots)$$
$$Z_1(q) = q^{-\frac{55}{24} + \frac{1}{10}}(8625q^{\frac{3}{2}} - 1138500q^{\frac{5}{2}} + 3777474000q^{7/2} + 3102750380125q^{9/2} \cdots)$$
$$Z_2(q) = q^{-\frac{55}{24} + \frac{2}{5}}(-1218500q^2 + 441969250q^3 + 953712511250q^4 + 217571250023750q^5 \cdots)$$
(3.13)

This is surprisingly close to (3.7) obtained by the "naive" counting of D4-D2-D0 bound states. Even more surprisingly, the first three coefficients in $Z_0(q)$ and the first two coefficients in $Z_1(q), Z_2(q)$ exactly match the answer obtained from the dilute gas approximation in AdS_3 (3.8)! One may also view this result as an exact "prediction" of the Gromov-Witten invariants of genus 0, degree 2 and 3, from the modular invariance of the modified elliptic genus.

Note that the fourth term in the q-expansion of $Z_0^{c.r.}(q)$ in (3.8) differ slightly from the one in (3.13). This mismatch might be due to corrections to the dilute gas approximation, or equivalently, the chiral ring relations we have assumed. It would be nice to understand this precisely.

We conjecture that (3.13) gives the exact modified elliptic genus of the M5-brane CFT on the quintic with $p = 1$. In closed form, using the modular forms defined in the appendix, we can write (equivalent to (3.13))

$$Z(\tau, \overline{\tau}, y) = \frac{1}{16\eta^{55}} \left[(20E_4^6 + 24500E_4^3 \Delta - 10703200\Delta^2)P_0(\tau, \overline{\tau}, y) \right. \tag{3.14}$$
$$\left. + (-225E_4^5 + 167375E_4^2 \Delta)P_1(\tau, \overline{\tau}, y) + (125E_4^4 - 89875E_4 \Delta)P_2(\tau, \overline{\tau}, y) \right]$$

where $\Delta = \eta^{24}$, η, E_4, E_6 are understood to be functions of τ.

4. The Z$_5$ quotient

Now let us consider a different example, with the Calabi-Yau X being the Fermat quintic $\sum_{i=1}^{5} x_i^5 = 0$ modded out by the freely acting symmetry $\mathbf{Z}_5$, generated by $x_i \mapsto \omega^i x_i$ where $\omega = e^{2\pi i/5}$. This CY space has intersection form $6D_{111} = 1$ instead of 5 for the quintic. We shall consider the M5-brane wrapped on the 4-cycle of charge $p = 1$ and $p = 2$ respectively.

4.1 $p = 1$

A $p = 1$ divisor $P_1 \subset X$ takes the form $x^i = 0$. Unlike the case of quintic, now P_1 is rigid, and there are five of them. $\chi(P) = 11$, $J \cdot J = 1$. There is only one term in the expression (2.9) for the modified elliptic genus $Z(\tau, \overline{\tau}, y)$. In fact, it is completely fixed by its modular weight $(-\frac{3}{2}, \frac{1}{2})$ and its polar term $5q^{-\frac{11}{24}}$,

$$Z_1(\tau, \overline{\tau}, y) = 5\eta(\tau)^{-11} E_4(\tau) \theta_1(\overline{\tau}, y). \tag{4.1}$$

Note that $E_4(\tau)$ is also the theta function of the E_8 root lattice Γ_8. This result admits a very simple explanation. The factor η^{-11} comes from the partition function of D0 pointlike instantons on P_1, which counts the Euler character of the Hilbert scheme of points on P_1. $E_4(\tau) = \theta_{\Gamma_8}(\tau)$ counts the various way of dissolving D0-branes into fluxes $F \in H^2(P_1, \mathbf{Z})$ such that $F \cdot J = 0$. In fact, we have the decomposition

$$H^2(P_1, \mathbf{Z}) = \{\alpha - (\alpha \cdot J)J\} \oplus \mathbf{Z}J \tag{4.2}$$

The anti-self-dual part of the lattice is even, hence must be $-\Gamma_8$.

4.2 $p = 2$

In this case, the degree 2 divisor P_2 can be defined by polynomial of the form $ax_1x_4 + bx_2x_3 + cx_5^2 = 0$ and four other similar quadratic polynomials. $\chi(P_2) = 28$, $J \cdot J = 2$. There is no Freed-Witten anomaly in this case. We can write the modified elliptic genus in the form

$$Z_2(\tau, \overline{\tau}, y) = Z_0(\tau)\theta_3(2\overline{\tau}, y) + Z_1(\tau)\theta_2(2\overline{\tau}, y) \tag{4.3}$$

where $Z_0(\tau) = q^{-\frac{28}{24}}(a_0 + a_1 q + \cdots)$, $Z_1(\tau) = q^{-\frac{28}{24} + \frac{1}{4}}(b_0 + b_1 q + \cdots)$. a_0, a_1, b_0 are the only polar coefficients. It is again possible to write a basis for the exact modular vectors. Although, we have not been able to count higher coefficients directly from the classical moduli space of the D4-D2-D0 bound states, and hence cannot check them against the constraints from the modular invariance of $Z_2(\tau, \overline{\tau}, y)$.

5. Conclusion

We see from the very basic example, the $p = 1$ M5-brane on the quintic, that modular invariance imposes powerful constraints on the degeneracy of BPS states, which encodes highly nontrivial relations of enumerative geometric invariants. In fact, the coefficients of

the M5-brane elliptic genus can be thought of a class of new geometric invariants, which generalizes Gromov-Witten invariants. The M5-brane elliptic genus also gives a natural way of associating modular forms with (M-theory) attractor Calabi-Yau threefolds, which might or not have interesting connections to the modularity of arithmetic algebraic varieties.

Let us list a few problems to be studied subsequently:

- Going beyond the dilute gas approximation in counting chiral primary states in $AdS_3 \times S^2 \times X$. In particular, it would be nice to understand the chiral ring relations, say from the sigma model description of the $(0,4)$ CFT.

- To understand the singularities in the classical moduli space of D4-brane with fluxes, and their contributions to the BPS D4-D2-D0 bound states. This would allow a general definition of the relevant enumerative geometric invariants.

- Extending our counting of BPS states to M5-branes of higher degrees, i.e. $p > 1$, on the quintic; as well as to other Calabi-Yau manifolds. In particular, it would be nice to count the BPS states on the $\mathbf{Z}_5$ quotient of the Fermat quintic and compare to the modular property of the $p = 2$ elliptic genus. We expect the fragmentation of the BPS states to play a role here.

Acknowledgments

A. A basis of modular vectors

In this appendix we will construct explicitly a basis of modular vector of weight $-\frac{3}{2}$ in the same representation of $Z_k(\tau)$ (appearing in the elliptic genus of M5-branes wrapped on the hyperplane section in the quintic) and of the same polar terms. Alternatively, we can construct a basis for the modular form $Z(\tau, \overline{\tau}, y)$ directly. Let us start with the theta functions relevant for the right-moving sector of the $p = 1$ elliptic genus for the quintic are

$$\Theta_k(\overline{\tau}, y) = \sum_{n \in \mathbf{Z}} q^{\frac{1}{10}(5n+k+\frac{5}{2})^2} z^{5n+k+\frac{5}{2}} (-1)^{n+k} \tag{A.1}$$

where $q = e^{2\pi i \tau}$, $z = e^{2\pi i y}$.

From unitarity of the S and T matrices, it follows that

$$\sum_{k=0}^{4} \Theta_k(\tau, s) \Theta_k(\overline{\tau}, y) \tag{A.2}$$

transforms as a weak Jacobi form of weight $(\frac{1}{2}, \frac{1}{2})$.

Specializing (A.2) to $s = \frac{\tau}{2}$, $s = \frac{\tau+1}{2}$, $s = \frac{1}{2}$ gives rise to three basic functions,

$$S_2(\tau,\overline{\tau},y) = e^{5\pi i\tau/4} \sum_{k=0}^{4} \Theta_k\left(\tau,\frac{\tau}{2}\right)\Theta_k(\overline{\tau},y)$$

$$S_3(\tau,\overline{\tau},y) = ie^{5\pi i\tau/4} \sum_{k=0}^{4} \Theta_k\left(\tau,\frac{\tau+1}{2}\right)\Theta_k(\overline{\tau},y)$$

$$S_4(\tau,\overline{\tau},y) = i\sum_{k=0}^{4} \Theta_k\left(\tau,\frac{1}{2}\right)\Theta_k(\overline{\tau},y) \tag{A.3}$$

They transform under S and T as

$$S_4 \xleftrightarrow{S} S_2 \xleftrightarrow{T} S_3 \tag{A.4}$$

whereas S_4 and S_3 transform to themselves under T and S, respectively. (S_2, S_3, S_4) transform in exactly the same way as the Jacobi theta functions $(\theta_2, \theta_3, \theta_4)$, which are also the specialization of the weak Jacobi form $\theta_1(\tau, s)$ at $s = \frac{1}{2}, s = \frac{\tau+1}{2}, s = \frac{\tau}{2}$, multiplied by prefactors similar to the ones in (A.3).

Hence GSO projection can be used to make modular invariant expressions. In particular,

$$P_n(\tau,\overline{\tau},y) = \sum_{i=2}^{4} \theta_i(\tau)^{8n+3} S_i(\tau,\overline{\tau},y) \tag{A.5}$$

is a weak Jacobi form, of weight $(4n+2, \frac{1}{2})$.

The space of all possible weak Jacobi forms of the form

$$\sum_{k=0}^{4} Z_k(\tau)\Theta_k(\overline{\tau}, z) \tag{A.6}$$

with the appropriate polar terms for Z_k as described in section 3 is a module with coefficients being holomorphic modular forms. The three simplest expressions P_0, P_1, P_2 are a basis for this module.

A complete set of possible Jacobi forms in this module with a q expansion starting at $q^{-\frac{55}{24}}$ and of weight $(-\frac{3}{2}, \frac{1}{2})$ is

$$\frac{E_4(\tau)^6 P_0}{\eta(\tau)^{55}}, \frac{E_4(\tau)^3 P_0}{\eta(\tau)^{31}}, \frac{P_0}{\eta(\tau)^7}, \frac{E_4(\tau)^5 P_1}{\eta(\tau)^{55}}, \frac{E_4(\tau)^2 P_0}{\eta(\tau)^{31}}, \frac{E_4(\tau)^4 P_2}{\eta(\tau)^{55}}, \frac{E_4(\tau) P_2}{\eta(\tau)^{31}} \tag{A.7}$$

It is convenient to write a basis for the 5-dimensional modular vector $Z_k(\tau)$, instead of the function (A.6). This basis can be extracted from (A.7). Alternatively (and equivalently), one can directly construct three holomorphic modular vectors, $A_k(\tau)$, $B_k(\tau)$ and $C_k(\tau)$, which transform in the same modular representation as $Z_k(\tau)$, but have weight 2,

4 and 6. One can start by defining

$$A_0(\tau) = \sum_{i=2}^{4} \theta_i(\tau)^3 \theta_i(5\tau),$$

$$B_0(\tau) = \sum_{i=2}^{4} \theta_i(\tau)^7 \theta_i(5\tau),$$

$$C_0(\tau) = \sum_{i=2}^{4} \theta_i(\tau)^{11} \theta_i(5\tau). \qquad (A.8)$$

The rest of A_k, B_k, C_k ($k = 1, \ldots, 4$) can be obtained by modular transforms of (A.8). One can check that the resulting functions indeed form 5-dimensional modular representations, which may not be immediately obvious by this construction. Knowing A_k, B_k, C_k, one can then multiply them by $\eta(\tau)^{-55}$ times weight $24, 22, 20$ holomorphic forms respectively, to obtain a basis for $Z_k(\tau)$, as shown in section 3.3.

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